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MINIMUM SENSITIVITY OPTIMAL CONTROL  
FOR  
NONLINEAR SYSTEMS

By

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## TABLE OF CONTENTS

	Page
LIST OF TABLES . . . . .	v
LIST OF FIGURES . . . . .	vi
ABSTRACT . . . . .	vii
CHAPTER I - INTRODUCTION . . . . .	1
1.1 Motivation . . . . .	1
1.2 State of Art . . . . .	2
1.2.1 Relaxed Variational Problems . . . . .	2
1.2.2 Minimum Sensitivity Optimal Control . . . . .	3
1.3 Statement of the Problem . . . . .	4
1.4 Organization of the Thesis . . . . .	6
CHAPTER II - RELAXED SOLUTIONS IN PROBLEMS OF OPTIMAL CONTROL .	8
2.1 Introduction . . . . .	8
2.2 Relaxed Solutions . . . . .	8
2.3 The Algorithm for Constructing a Suboptimal Control .	11
2.4 Solving the Relaxed Problem . . . . .	12
2.5 Relation Between Existence of an Optimal Solution and Singular Solution of the Relaxed Problem . . .	14
2.6 Example 2.2 . . . . .	15
CHAPTER III - SENSITIVITY IN CONTROL THEORY . . . . .	27
3.1 Introduction . . . . .	27
3.2 Sensitivity Considerations in Classical Control Theory . . . . .	27
3.3 Sensitivity of Variable Structure Systems . . . . .	29
3.4 Definitions of Sensitivity in Optimal Control . . . .	30
3.5 Discussion of an Example Given by Holtzman and Horing . . . . .	31
CHAPTER IV - EXAMPLES OF MINIMUM SENSITIVITY OPTIMAL CONTROL .	34
Example 4.1 . . . . .	34
Example 4.2 . . . . .	43

	Page
CHAPTER V - SUMMARY AND CONCLUSIONS . . . . .	58
5.1 Summary . . . . .	58
5.2 Suggestions for Further Work . . . . .	58
LIST OF REFERENCES . . . . .	60
VITA . . . . .	62

## LIST OF TABLES

Table		Page
4.1	Chattering Control and Trajectory for Example 4.1 when $g = v^2 (1-ah)^{4.3} - b - 100 (\theta - \theta_N)$ . . . . .	41
4.2	Feedback Control and Trajectory for Example 4.1 when $g = v^2 (1-ah)^{4.3} - b$ . . . . .	42
4.3	Minimax Solution for Example 4.2 . . . . .	56

## LIST OF FIGURES

Figure		Page
2.1	Sub-Optimal Solution for Example 2.1 . . . . .	21
3.1	Relay Feedback System . . . . .	29
3.2	PI as a Function of Sensitivity . . . . .	33
4.1	Relaxed Control and Sub-Optimal Control for Example 4.2 . . . . .	54
4.2	Comparison of $x_{2_{\text{relaxed}}}$ and $x_2$ for 10 Switchings . . .	55

## ABSTRACT

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Relaxed Variational Techniques are applied to a minimum sensitivity control problem. Sensitivity of a trajectory is minimized to perturbations in initial conditions. Rather than using the optimal control that does indeed exist and that satisfies the final conditions exactly, a sub-optimal control is used that transfers the system from the given initial state to an arbitrarily small neighborhood of the given final state, and that results in a considerably better performance than the optimal solution. The sub-optimal control is constructed using the optimal controls of the relaxed problem.

It is demonstrated by an example that a sub-optimal chattering control obtained for a minimum time problem can be made a function of the states of the system and thus lower sensitivity to perturbations in initial states is achieved as well as minimum time.

A relation is shown to exist between relaxed problems and singular control. It is shown that a problem that does not possess an optimal solution, but satisfies some general assumptions, has a singular relaxed solution.

## CHAPTER ONE

### INTRODUCTION

#### 1.1 Motivation

In optimal control problems the Maximum Principle gives necessary conditions for the optimal control and trajectories [Pontryagin et al, ref. 1, Berkovitz ref. 2]. The differential equations representing the necessary conditions are derived on the assumption that an optimal solution does indeed exist. However, except for very special classes of problems, existence of optimal solutions is not guaranteed. It was shown recently [Warga ref. 3], under some general assumptions that for control problems that do not possess an optimal solution it is possible to construct a sub-optimal solution. Denote the lower bound on the performance index by  $N$ , then the sub-optimal solution will result in a performance index as close to  $N$  as desired. The sub-optimal control will transfer the system from the given initial state to an arbitrarily small neighborhood of the given final state. Furthermore it was shown that if final conditions are specified for the system at the terminal time and an optimal control does indeed exist for the problem, it might be possible by relaxing the end conditions to obtain a sub-optimal control that would result in a considerably better performance than the optimal.

A general class of non-linear systems where the control is coupled with the state variables is treated here. It is desired to transfer the given initial state to the given final state minimizing an expression that describes sensitivity of the response of the system with respect to perturbations in initial conditions.

For such a problem it is possible, by relaxing the final conditions for the problem, to obtain a substantial improvement in the performance of the system. Rather than using the optimal control that does indeed exist and that satisfies the final conditions exactly, one can use a sub-optimal control that would transfer the system from the given initial state to an arbitrarily small neighborhood of the given final state, and that results in a considerably better performance than the optimal solution. This approach is beneficial for problems where it is not essential to satisfy the final conditions exactly.

It is shown that the approach mentioned above applies to systems where "Stabilizing Signal" [Oldenburger ref. 5] can be used for stabilization or improvement of system performance.

## 1.2 State of Art

### 1.2.1 Relaxed Variational Problems

Relaxed variational problems were introduced by Gamkrelidze [ref. 6] and were discussed in detail by Warga [ref. 3] and by Krotov [ref. 7]. The discussions are mainly theoretical and are restricted to the proof of theorems on relaxed problems.

One application of relaxed variational problems to an engineering problem was given by Gurman [ref. 8]. The problem discussed is that of a coasting airplane that is to be transferred from initial to final

conditions in minimum time. It is shown that in order to satisfy a certain constraint on the state variables, a chattering control is the accepted sub-optimal control.

To the best of the author's knowledge no other applications were made of relaxed controls to engineering problems.

#### 1.2.2 Minimum Sensitivity Optimal Control

Classical sensitivity has recently been employed by Dorato [ref. 9] who discussed the sensitivity of the performance index in the optimal control problem with respect to plant parameter variations. A method was outlined for computing the performance index sensitivity functions. This will be discussed in Chapter Three.

Further results were obtained by Pagurek [ref. 10] through the use of the Hamilton-Jacobi equation. For linear systems with quadratic performance indices it was shown that the performance index sensitivity functions for the open loop case and for the closed loop case are identical. Witsenhausen [ref. 11] extended Pagurek's results to non-linear problems. It was later shown by Sinha and Atluri [ref. 12] that Pagurek obtained his results by considering infinitesimal variations in the system parameters. They showed further that if small but finite variations are used, the closed loop configuration might be superior.

A new definition of relative sensitivity in optimal control problems was introduced by Rohrer and Sobral [ref. 13].

Sensitivity of terminal conditions to parameter variations was examined by Holtzman and Horing [ref. 14]. Inclusion of specification

of sensitivity for certain parameters in open loop design was demonstrated. It is shown in Chapter 3 of this thesis that the results obtained by Holtzman and Horing are misleading because of the fact that sensitivity was defined for finite variations in the parameter, and replaced by infinitesimal variations.

Rissanen [ref. 15] deals with the problem of evaluating the extent to which system parameters may be changed and still guarantee that system performance will remain within a specified limit.

### 1.3 Statement of the Problem

We consider the system of equations

$$\dot{x} = f(x, t, u) \quad (1.1)$$

with initial conditions

$$x(0) = x^0 \quad (1.2)$$

where  $x$  is an  $(n-1)$  vector

$$x = \text{col } [x_1(t), x_2(t), \dots, x_{n-1}(t)]$$

$f$  is an  $(n-1)$  vector and  $u$  is a scalar function of time. It is desired to find a function  $u(t)$ , called the control, belonging to a class  $U$  of measurable functions such that the initial point  $x^0$  is transferred by system (1.1) to some terminal manifold  $D$  which may be described by the equations

$$g_j(x) = 0 \quad j = 1, 2, \dots, m \quad m \leq n-1 \quad (1.3)$$

in the fixed time duration  $t = T$ . We shall assume that there exists at least one function  $u(t)$  in the class  $U$  of measurable functions which transfers the initial point to  $D$  in the given time  $T$ . Such a control will be called "admissible".

We shall define the sensitivity  $S$  of the plant to initial

disturbances by the relation

$$S = S(t) = \int_0^t f_0(x, t, u) dt \quad (1.4)$$

where  $f_0(x, t, u)$  is a scalar function. We shall denote  $S(t)$  by  $x_0(t)$  and add the equation

$$\dot{x}_0 = f_0(x, t, u) \quad (1.5)$$

to system (1.1).

It is desired to minimize  $x_0(T)$ . Equations (1.1) may represent plant equations, constraint equations and perturbation equations.

The vector  $f$  as well as the scalar  $f_0$  may be nonlinear functions of any or all of their variables. In this case it may occur that no solution to this problem exists. The performance index  $x_0(T)$  will be assumed to have a lower bound  $J$ . However, it may happen that there is no admissible control for which the performance index is  $J$ . However, the value  $J$  may be approximated as closely as desired by appropriate choice of  $u(t)$  under rather general conditions, as will be demonstrated below. It will also be shown that in many cases a performance index which is considerably smaller than  $J$  can be obtained by satisfying equations (1.3) only approximately. This will be done by designing a control  $u = u(t)$  for which  $u(t)$  assumes only a finite number of values.

A control belonging to  $U$  will be termed "suboptimal" if it transfers the initial state to a small neighborhood of the terminal manifold while reducing the sensitivity  $x_0(T)$  to a value close to  $J$  or lower. An  $\epsilon$  neighborhood of the manifold  $D$  is the set of points that contains  $D$  and all points that are at most at a distance  $\epsilon$  from the manifold.

We shall make the following continuity and boundedness assumptions

throughout this thesis. There exists an open set  $V$  in Euclidean  $n$  space,  $E_n$ , a compact set  $A \subset V$ , and a closed set  $B$  contained in  $A$  such that  $D \subset B$ , and for all  $u$  in  $U$  and  $0 \leq t \leq T$ , the following hold.

- I.  $f(x, t, u)$  and  $f_0(x, t, u)$  are continuous in  $t$  uniformly in  $x$  and  $u$  for all  $x$  in  $V$ .
- II.  $\|f(x, t, u) - f(y, t, u)\| \leq k \|x - y\|$   
and  $|f_0(x, t, u) - f_0(y, t, u)| \leq k \|x - y\|$   
for all  $x, y$  in  $V$ , where double bars indicate Euclidean norm and single bars mean absolute value.
- III. There exists a positive constant  $M$  such that  
$$\|f(x, t, u)\| + |f_0(x, t, u)| \leq M$$
for all  $x$  in  $V$ .
- IV.  $f(x, t, u)$  and  $f_0(x, t, u)$  is continuous in  $(x, t)$  uniformly in  $u$  for all  $x$  in  $A$ .
- V.  $f(x, t, u)$  and  $f_0(x, t, u)$  is continuous in  $u$  for all  $x$  in  $V$ .

The above assumptions guarantee that our methods apply. In particular, assumption III guarantees that  $x_0(T)$  has a finite lower bound and that no solution has an escape time  $\tau$  less than  $T$ . A solution  $x(t)$  of a differential system  $\dot{x} = \phi(x, t)$ , is said to have an escape time  $\tau > 0$  if  $\lim_{t \rightarrow \tau_-} x(t) = +\infty$ . The other assumptions guarantee existence and uniqueness of solutions in the appropriate regions to the system of differential equations. We shall say more about conditions I-V later.

#### 1.4 Organization of the Thesis

Chapter 2 begins with a discussion of Relaxed Variational Problems. We show that when problems that do not possess optimal solutions are relaxed, they become singular. Sensitivity considerations in classical

and optimal control are included in Chapter 3. Zero sensitivity for systems in the chattering state is discussed. Some examples given in the literature are discussed. It is shown that definitions of sensitivity for infinitesimal variations may lead to erroneous results. Minimum sensitivity optimal control is discussed in Chapter 4, and two examples are presented. In the first example a sub-optimal open loop chattering control is made a function of the states and thus becomes a feedback control. In the other example relaxed variational techniques are applied to the construction of a sub-optimal control to minimize sensitivity of a trajectory to perturbations in initial conditions. Suggestions and recommendations for future work are given in Chapter 5.

## CHAPTER TWO

### RELAXED SOLUTIONS IN PROBLEMS OF OPTIMAL CONTROL

#### 2.1 Introduction

Relaxed solutions in problems of optimal control are discussed. An algorithm is given for constructing a sub-optimal control. The relation between existence of an optimal solution and singular solution of the relaxed problem is discussed. An example is given of an optimization problem that does not possess an optimal solution, whereas the relaxed problem is shown to have a singular solution.

#### 2.2 Relaxed Solutions

We augment  $x_0$  to  $x$  and thus redefine  $x$  to be

$$x = \text{col} (x_0, x_1, \dots, x_{n-1})$$

We also augment  $f_0$  to  $f$  and thus redefine

$$f = \text{col} (f_0, f_1, \dots, f_{n-1})$$

The problem stated in section 1.3 of this thesis will be called the "original problem". Consider the set of points  $y$  in  $n$  space given by

$$F(x,t) = \{y \mid y = f(x,t,u), u \in U\}$$

$F(x,t)$  is defined for each fixed  $(x,t)$  with  $u(t)$  varying over the set  $U$ . Then condition (1.1) of Chapter 1 of this thesis can be written as

$$\dot{x}(t) \in F[x(t), t] \quad (2.1)$$

The set  $F(x,t)$  is thus the set of all permissible values of  $\dot{x}$  while passing through the point  $x$  at the time  $t$ .

We shall introduce the "relaxed problem" for problem (1.1). Let  $G(x, t)$  be the convex closure, or the convex hull of the closure, of  $F(x, t)$ . The relaxed problem consists in minimizing  $x_0(T)$  subject to the assumptions in section 1.3 of this thesis and

$$\dot{x} \in G[x(t), t] \quad (2.2)$$

The problem is relaxed in the sense that the permissible set of choices of  $\dot{x}(t)$  is enlarged from  $F(x, t)$  to  $G(x, t)$ .

Define the relaxed system of equations for equations (1.1).

$$\dot{x} = \sum_{i=1}^{n+1} \alpha_i(t) f(x, t, u_i) \quad (2.3)$$

with the same initial conditions as for the original problem,  $x(0) = x^0$ .

The scalar controls  $\alpha_i^*(t)$  and  $u_i^*(t)$ ,  $i = 1, \dots, n+1$  are to be found such that

$$g_j[x^*(T)] = 0, \quad j = 1, \dots, m \quad m \leq n-1$$

where  $x = x^*(t)$  is the trajectory of (2.3), when  $\alpha_j = \alpha_j^*(t)$  and  $u_j = u_j^*(t)$ ,  $j = 1, \dots, n+1$ , and

$$x_0^*(T) = \min_{\substack{\alpha_i \in C \\ u \in U}} \{x_0(T)\} \quad (2.4)$$

where  $C$  is a subset of the set of measurable functions,

$$\alpha_i(t) \geq 0 \quad \sum_{i=1}^{n+1} \alpha_i(t) \equiv 1$$

and each  $g_j(x)$  is independent of  $x_0$ .

A "relaxed admissible curve" is defined as any absolutely continuous vector function  $x(t)$  satisfying conditions (2.3), (1.2) and (1.3).

The following theorems are given by Warga [ref. 3].

**Theorem I.** If the vector functions  $f(x,t,u)$  of equation (1.1) and the scalar function  $f_0$  of equation (1.5) satisfy conditions I-V of section 1.3 of this thesis, then every absolutely continuous curve  $x(t)$ , satisfying equation (2.3) is the uniform limit of curves  $x_N(t)$ ,  $N = 1, 2, \dots$ , satisfying differential equations (1.1) and (1.5) and such that  $x_N(0) = x(0)$ ,  $N = 1, 2, \dots$ .

**Theorem II.** If  $f(x,t,u)$  and  $f_0(x,t,u)$  satisfy the assumptions of Theorem I and if there exists a relaxed admissible curve, then there exists a relaxed minimizing curve.

**Theorem III.** Let  $f(x,t,u)$  and  $f_0(x,t,u)$  satisfy the assumptions of Theorem I and let  $B = E_n$ . Assume that there exists an original minimizing curve  $x(t)$ . Then  $x(t)$  is also a relaxed minimizing curve.

Theorem III does not, in general, remain valid when the assumption  $B = E_n$  is dropped. This is demonstrated by the following counter-example, example 2.1, [Warga, ref. 3], and will be demonstrated later again by example 4.2.

**Example 2.1**

$$\begin{aligned} \dot{x}_1 &= x_2^2 - u^2 & x_1(0) &= 0 \\ \dot{x}_2 &= u & x_2(0) &= 0 \\ \dot{x}_3 &= x_2^4 & x_3(0) &= 0 \quad x_3(T) = 0 \end{aligned} \tag{2.5}$$

Minimize  $x_1(T)$ ,  $T > 0$ , subject to  $|u| \leq 1$ . There exists an original minimizing curve  $x_1(t) = x_2(t) = x_3(t) = 0$ .

Since  $x_3(0) = x_3(T) = 0$  and  $\dot{x}_3 = x_2^4 \geq 0$  it follows that  $x_2(t) \equiv 0$  and therefore  $u(t) = 0$ .

The relaxed problem is

$$\begin{aligned}
\dot{x}_1 &= x_2^2 - \alpha u_1^2 - (1-\alpha) u_2^2 & x_1(0) &= 0 \\
\dot{x}_2 &= \alpha u_1 + (1-\alpha) u_2 & x_2(0) &= 0 \\
\dot{x}_3 &= x_2^4 & x_3(0) &= 0 ; x_3(T) = 0
\end{aligned} \tag{2.6}$$

It is necessary to consider here only two  $\alpha$ 's instead of four, since one is eliminated because the set of admissible controls  $U$  is connected and the second is eliminated since the control  $u$  appears in only two of the equations (2.5). Consider a special relaxed admissible curve with the control  $u_1 = 1$ ,  $u_2 = -1$  and  $\alpha = 1/2$ . Equations (2.6) become

$$\begin{aligned}
\dot{x}_1 &= x_2^2 - 1 & x_1(0) &= 0 \\
\dot{x}_2 &= 0 & x_2(0) &= 0 \\
\dot{x}_3 &= x_2^4 & x_3(0) &= 0 ; x_3(T) = 0
\end{aligned} \tag{2.7}$$

It follows that  $x_1(t) = -t$ ,  $x_2(t) = 0$ ,  $x_3(t) = 0$ ,  $x_1(T) = -T$ . Since  $x_3(T) = 0$  it follows the solution is admissible. It is in fact a relaxed minimizing curve since  $\dot{x}_1 = x_2^2 - u^2 \geq -1$  implies  $x_1(T) \geq -T$ . Thus, there exists an original minimizing curve which is not a relaxed minimizing curve. Furthermore,  $x_1^{\text{relaxed}}(T) < x_1^{\text{original}}(T)$

### 2.3 The Algorithm for Constructing a Suboptimal Control

It is shown in reference [3] that it is possible to construct a suboptimal control using the optimal controls of the relaxed problem. An algorithm to do that is provided by Gamkrelidze in reference [6]. The algorithm used in this work is similar to the one given by Gamkrelidze and is described here.

It is assumed that the optimal controls  $u_1^*(t)$  and  $\alpha_1^*(t)$ ,

$i = 1, \dots, n+1$  were found for the relaxed problem. The suboptimal control for the original problem is constructed as follows. Divide the time interval  $[0, T]$  into an arbitrary number  $N$  of equal subintervals. Denote the beginning of the first interval by  $t_0$ , the end of the first interval by  $t_1$ , end of second interval by  $t_2$  and end of  $N$ th interval by  $t_N$ . Every interval  $[t_k, t_{k+1}]$  is divided into  $(n+1)$  subintervals; the length of the  $j$ th subinterval shall be  $\alpha^*(t_k) [t_{k+1} - t_k]$  for  $j = 1, \dots, n+1$ . At the  $j$ th subinterval the control  $u_j^*(t_k)$  is applied. As  $N$  approaches infinity, the trajectory described by the original system will approach the optimal trajectory of the relaxed system and the performance index of the original problem will approach the performance index of the relaxed problem.

#### 2.4 Solving the Relaxed Problem

Given a plant

$$\dot{x} = f(x, t, u) \quad (2.8)$$

with initial conditions

$$x(0) = x^0 \quad (2.9)$$

and final conditions given by the terminal manifold

$$g_j(x) = 0 \quad j = 1, \dots, m \quad m \leq n-1 \quad (2.10)$$

where  $g_j(x)$  are independent of  $x_0$ , and  $x = \text{col}(x_0, x_1, \dots, x_{n-1})$ .

Assume that the control  $u(t)$  is taken from a set  $U$  as before. Find

$u^*(t)$ , in  $U$ ,  $0 \leq t \leq T$ , that satisfies

$$x_0^*(T) = \min_{u \in U} \{x_0(T)\} \quad (2.11)$$

where the terminal time  $T$  is a constant.

When a solution exists to (2.8), (2.9), (2.10) and (2.11) the Maximum Principle gives necessary conditions for the optimal control and

trajectory. Define the Hamiltonian

$$H = \langle \lambda, f(x, t, u) \rangle \quad (2.12)$$

where the symbol  $\langle , \rangle$  means dot product, then

$$\dot{x} = H_{\lambda} \quad (2.13)$$

and

$$\dot{\lambda} = -H_x \quad (2.14)$$

where  $\lambda = \text{col} (\lambda_0, \lambda_1, \dots, \lambda_{n-1})$  is the adjoint state vector.

The transversality condition

$$H dt + \langle \lambda, dx \rangle = 0 \quad (2.15)$$

furnishes boundary conditions for  $\lambda$ . According to the Maximum Principle the optimal control  $u^*(t)$  must minimize the Hamiltonian at all times  $0 \leq t \leq T$ .

An optimal solution for (2.8), (2.9), (2.10) and (2.11) may not exist. If a solution does exist a suboptimal solution might result in better performance than the optimal solution. In both cases relaxed solutions must be considered.

Define the relaxed equations of (2.8)

$$\dot{x} = \sum_{j=1}^{n+1} \alpha_j(t) f(x, t, u_j) = G(x, t, \alpha_j, u_j) \quad (2.16)$$

$$\sum_{j=1}^{n+1} \alpha_j(t) = 1, \quad \alpha_j(t) \geq 0$$

with initial conditions (2.9) and final conditions (2.10) where  $x$  is now the augmented vector  $\text{col} (x_0, x_1, \dots, x_{n-1})$ . Find  $u_j^*(t), \alpha_j^*(t)$  that satisfy

$$x_0(T) = \text{Min}_{\alpha_j, u_j} \{x_0(T)\} \quad (2.17)$$

To solve (2.16), (2.17), (2.9) and (2.10) define the Hamiltonian

$$H = \langle \lambda, G(x, t, \alpha_j, u_j) \rangle \quad (2.18)$$

then

$$\dot{x} = G(x, t, \alpha_j, u_j) \quad (2.19)$$

$$\dot{\lambda} = -H_x$$

According to the Maximum Principle the following must be satisfied,

if  $\lambda_0 > 0$

$$H(\lambda, x, t, \alpha_j^*, u_j^*) = \min_{\alpha_j, u_j} \{H(\lambda, x, t, \alpha_j, u_j)\} \quad j=1, \dots, n+1 \quad (2.20)$$

It follows from (2.12), (2.16), (2.18) and (2.20) that every one of the controls  $u_j^*(t)$  also minimizes the Hamiltonian of the original problem (2.12) evaluated along the optimal trajectory.

## 2.5 Relation Between Existence of an Optimal Solution and Singular Solution of the Relaxed Problem

In many applications it is possible to express  $u_j^*$  as a function of  $x$  and  $\lambda$  and when substituted into (2.16), (2.19) a new problem arises which is linear in the controls  $\alpha_j$ .

It is known from the theory of optimal control that an optimization problem which is linear in the control can have either a singular solution or a solution with the control on the boundary.

Suppose a system and a performance index are described by a set of differential equations linear in the control. From the necessary conditions for optimality it is clear that for minimizing the performance index, the optimal control  $u^*(T)$  has to assume values on the boundary of  $U$ , where  $u \in U$ . If the coefficient that multiplies  $u$  in the Hamiltonian vanishes for a certain time interval, the optimal control might take values inside  $U$ . The problem is then called singular. If one of the

$\alpha$ 's is identically equal to 1, on some time interval, that means that all the other  $\alpha$ 's are identically zero on this interval and the original problem has a solution on this time interval. The other alternative is that none of the  $\alpha_j$  is identically equal to 1 and a singular solution should be accepted for the relaxed problem. Thus the following theorem can be stated.

Theorem: The relaxed problem has a singular solution if the original problem does not have an optimal solution.

Proof: The theorem is proven by contradiction. Assume that the original problem does not possess an optimal solution. It is known from the sufficient conditions, described in section 1.3 of this thesis, that the relaxed problem does have an optimal solution. If the relaxed optimal controls  $\alpha_j(t)$ ,  $j = 1, \dots, n+1$  do indeed assume values on the boundary of  $C$  for the whole time interval  $[0, T]$ , then one of the  $\alpha$ 's, say  $\alpha_k$ ,  $1 \leq k \leq n+1$ , is identically equal to 1 and the other  $\alpha$ 's are identically equal to zero for every subinterval of  $[0, T]$ . But if this is the case then the relaxed problem is reduced to the original problem. Since it is known that the relaxed problem has an optimal solution, so would the original problem, but this is a contradiction to the basic assumption that the original problem does not have an optimal solution. The conclusion is that on at least some time interval that belongs to  $[0, T]$ , the relaxed controls  $\alpha_j$  assume values between zero and one,  $0 < \alpha_j < 1$ , and this case is singular by definition.

## 2.6 Example 2.2

The following example demonstrates that for an optimal control problem that does not have a solution, the relaxed problem has a singular

solution.

It is first shown that the given problem does not have an optimal solution. For clarity this problem is called problem A.

### Problem A

Consider

$$\dot{x}_0 = \frac{1}{2} x_1^2 + \sqrt{u} \quad x_0(0) = 0 \quad (2.21)$$

$$\dot{x}_1 = x_2 + u \quad x_1(0) = -2, x_1(T) = -1 \quad (2.22)$$

$$\dot{x}_2 = -u \quad x_2(0) = 1, x_2(T) = 0.5 \quad (2.23)$$

with the constraint on the control  $0 \leq u \leq 1$ .

It is desired to transfer the system state variables from the given initial conditions to the given final conditions and minimize  $x_0(T)$ , where the terminal time  $T$  is free.

The Hamiltonian for this problem is

$$H = \frac{1}{2} x_1^2 + \sqrt{u} + \lambda_1(x_2 + u) + \lambda_2(-u) \quad (2.24)$$

Now, if

$$\frac{\partial H}{\partial u} = \frac{1}{2} u^{-\frac{1}{2}} + \lambda_1 - \lambda_2 = 0 \quad (2.25)$$

then computing the second derivative of  $H$  with respect to  $u$ , one obtains

$$\frac{\partial^2 H}{\partial u^2} = -\frac{1}{4} u^{-\frac{3}{2}} < 0 \quad (2.26)$$

It is seen from (2.26) that if  $0 < u^*(t) \leq 1$ , then  $u^*$  furnishes a maximum, and not a minimum as required.

Therefore, the following can be stated. If an optimal control does indeed exist for problem A, it can assume only the values 0 or 1.

When the control is  $u = 1$ , one gets:

$$\dot{x}_1 = x_2 + 1 \quad (2.27)$$

$$\dot{x}_2 = -1 \quad (2.28)$$

from (2.28)

$$x_2 = -t + c_1 \quad (2.29)$$

and from (2.27)

$$\dot{x}_1 = -t + c_1 + 1 \quad (2.30)$$

$$x_1 = -\frac{t^2}{2} + (c_1 + 1)t + c_2$$

It follows that

$$x_1 = -\frac{1}{2}x_2^2 - x_2 + k \quad (2.31)$$

Equation (2.31) describes a set of parabolas.

For  $u = 0$  one gets

$$\dot{x}_1 = x_2 \quad (2.32)$$

$$\dot{x}_2 = 0 \quad (2.33)$$

From (2.33)

$$x_2 = \text{constant} \quad (2.34)$$

In order to show that problem A does not possess an optimal solution, problem B is introduced here.

Problem B

Consider

$$\dot{x}_0 = \frac{1}{2} x_1^2 \quad x_0(0) = 0 \quad (2.35)$$

$$\dot{x}_1 = x_2 + u \quad x_1(0) = -2; \quad x_1(T) = -1 \quad (2.36)$$

$$\dot{x}_2 = -u \quad x_2(0) = 1; \quad x_2(T) = 0.5 \quad (2.37)$$

with the same constraint on the control

$$0 \leq u \leq 1$$

It is shown by Johnson and Gibson [ref. 4] that the optimal control for problem B is  $u^*(t) = x_2^*(t)$ . Now it is shown by contradiction that problem A does not possess an optimal solution. Assume that problem A does indeed have an optimal solution. From (2.26) it is clear that  $u^*(t)$  would assume the values 0 or 1.

Let us compute the difference in the two performance indices (2.21) and (2.35) for problem A and problem B, respectively, for the same control  $u(t)$  out of the set of admissible controls.

Equation (2.21) is different from equation (2.35) at times when  $u(t) = 1$  and is the same as (2.35) when  $u(t) = 0$ .

From (2.23) it is seen that  $x_2 = \text{constant}$  when  $u \equiv 0$ . Also, when  $u \equiv 1$ , it is seen from (2.23) that

$$\dot{x}_2 = -1 \quad (2.38)$$

or

$$dx_2 = -dt \quad (2.39)$$

From equation (2.39) it is clear that the total time for which  $u(t) = 1$  is

$$x_2(0) - x_2(T) = 1 - 0.5 = 0.5 \quad (2.40)$$

It follows that for the assumed optimal control  $u^*(t)$ , or any other admissible control, that takes values of 0 or 1 only, the performance indices given by (2.21) and (2.35) differ by a constant equal to 0.5.

It follows that the assumed optimal control  $u^*(t)$  for problem A should be a candidate optimal solution also for problem B. However, it was shown by Johnson and Gibson [ref. 4] that the optimal control for problem B is singular and that  $u^*(t) = x_2^*(t)$ .

It remains to be shown for problem B that for any admissible control that only assumes values of 0 or 1, there is another admissible control that only assumes the values 0 or 1 that provides a performance index closer to the optimal performance index, the latter corresponding to the singular control.

In the following it is shown how to construct a control that assumes only the values 0 or 1 and which provides a performance index as close to the optimal as is desired. For that purpose, consider the following relaxed problem for problem B.

$$\dot{x}_0 = \frac{1}{2} x_1^2 \quad x_0(0) = 0 \quad (2.41)$$

$$\dot{x}_1 = x_2 + \alpha u_1 + (1-\alpha) u_2; \quad x_1(0) = -2; \quad x_1(T) = -1 \quad (2.42)$$

$$\dot{x}_2 = -\alpha u_1 - (1-\alpha) u_2; \quad x_2(0) = 1; \quad x_2(T) = 0.5 \quad (2.43)$$

Choose base controls

$$u_1 = 1$$

and

$$u_2 = 0$$

to get

$$\dot{x}_0 = \frac{1}{2} x_1^2 \quad x_0(0) = 0 \quad (2.44)$$

$$\dot{x}_1 = x_2 + \alpha \quad x_1(0) = -2 \quad ; \quad x_1(T) = -1 \quad (2.45)$$

$$\dot{x}_2 = -\alpha \quad x_2(0) = 1 \quad ; \quad x_2(T) = 0.5 \quad (2.46)$$

with the constraint  $0 \leq \alpha < 1$ .

According to Johnson and Gibson [ref. 4], the optimal control for (2.44), (2.45) and (2.46) is

$$\alpha^*(t) = x_2^*(t) \quad (2.47)$$

Let the time it takes to get from the initial to the final point with the singular control equal  $T_1$ . According to Warga [ref. 3] it is possible to construct a sub-optimal control for problem B consisting of  $u = 1$  and  $u = 0$  switching according to  $\alpha^*(t)$  given by (2.47). The control thus constructed will not necessarily satisfy the final conditions of the problem. However, one could get into  $S$  by using a sufficiently large number of switchings (see figure 2.1). The region  $S$  is bounded by two trajectories corresponding to  $u \equiv 0$  and two trajectories corresponding to  $u \equiv 1$ , the point  $D$  being its right vertex.

Starting from the initial point the sub-optimal control found in section 2.3, is used with  $N$  so large that the point  $(x_1, x_2)$  arrives within the region  $S$ . Once in region  $S$ , one can get to the final point with at most two switchings. The contribution of the last step to the performance index is negligible, since  $S$  can be chosen as small as desired.

It will now be shown that the performance index for the trajectory ABCD of figure 2.1 can be made as close to the optimal as desired.

According to Warga, the performance index along the trajectory AB is greater than the optimal by at most  $\epsilon_1$  where  $\epsilon_1 > 0$  is a pre-set

constant if only  $N$  is large enough. Thus,

$$PI_{AB} - PI_{opt.} \leq \epsilon_1$$

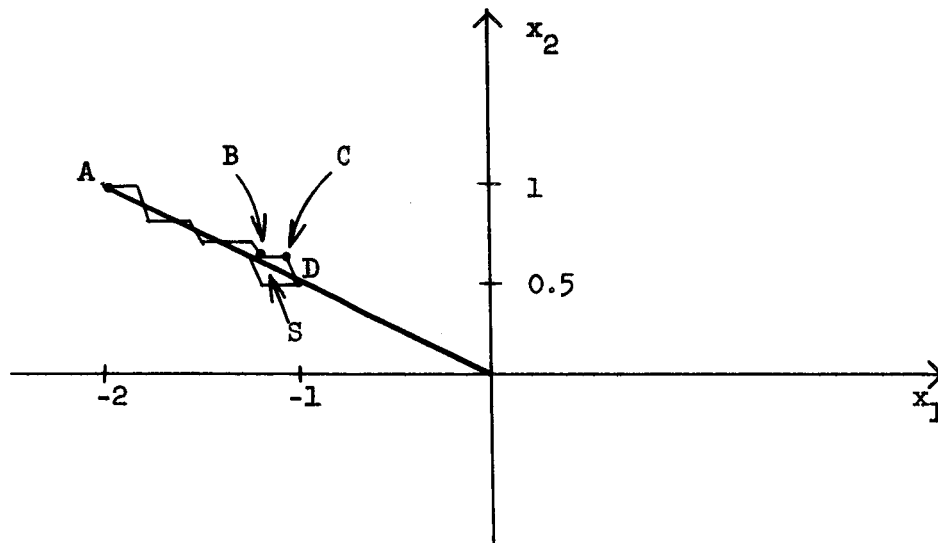


Figure 2.1. Sub-Optimal Solution for Example 2.1.

Warga's algorithm is applied to this problem. Divide  $T_1$  into  $N$  equal parts. The  $k$ th time interval starts at  $t = \frac{T_1}{N} (k-1)$  and ends at  $t = \frac{T_1}{N} k$ . For the  $k$ th interval, apply the control  $u = 1$  for time  $\tau$

where

$$\tau = \Delta \cdot x_2^*, \quad \Delta = \frac{T_1}{N}$$

and  $x_2^*$  is the expression for the singular optimal solution evaluated at the beginning of the interval i.e., at  $t = \frac{T_1}{N} \cdot (k-1)$ . For the rest of the interval  $\Delta$ , apply the control  $u = 0$ .  $N$  is chosen such that  $B$  is inside  $S$ , and  $PI_{AB} - PI_{opt} \leq \epsilon_1$ . Along the line  $BC$  the added term in the performance index is

$$\frac{1}{2} \int_{t_B}^{t_C} x_1^2 dt < \frac{1}{2} \int_{t_B}^{t_C} 2^2 dt = 2 \int_{t_B}^{t_C} dt \text{ where } t_B \text{ and } t_C \text{ are the}$$

times the state is at point B and point C, respectively.

From (2.32)

$$\dot{x}_1 = x_2 \geq 0.5$$

We can choose  $S = S_1$  so small that given  $\epsilon_3$

$$\frac{1}{2} \int_{t_B}^{t_C} x_1^2 dt < \epsilon_3$$

Along the line CD the added term in the performance index is

$$\frac{1}{2} \int_{t_C}^{t_D} x_1^2 dt < \frac{1}{2} \int_{t_C}^{t_D} 2^2 dt = 2 \int_{t_C}^{t_D} dt.$$

Since for the path CD

$$\dot{x}_2 = -1$$

it follows similarly that  $S = S_2$  can be chosen such that

$$\frac{1}{2} \int_{t_C}^{t_D} x_1^2 dt < \epsilon_4.$$

It follows that the time it takes to get from B to C and from C to D can be made as small as desired, by choosing S to be the smaller of the regions  $S_1$  or  $S_2$ .

Thus

$$PI_{ABCD} - PI_{opt} < \epsilon_1 + \epsilon_3 + \epsilon_4 \triangleq \epsilon$$

or

$$PI_{ABCD} - PI_{opt} < \epsilon$$

and the performance index along the path ABCD using an admissible control that assumes only the values 0 or 1 can be made as close to the optimal performance index as desired. If the state of the system does not fall

into the region  $S$  at time  $T_1$ , then it would be necessary to use the algorithm for less than  $T_1$  seconds to get to the point  $B$ , inside  $S$ .

Two cases should be considered.

$$\text{Case (I)} \quad 0 \leq PI_{AB} - PI_{opt} \leq \epsilon_1$$

In this case the previous argument holds.

$$\text{Case (II)} \quad PI_{AB} = PI_{opt} - \delta ; \quad \delta > 0$$

As before

$$PI_{BC} + PI_{CD} < \epsilon_3 + \epsilon_4$$

$$PI_{ABCD} - PI_{opt} = PI_{AB} - PI_{opt} + PI_{BC} + PI_{CD} < -\delta + \epsilon_3 + \epsilon_4$$

$$\text{Also } PI_{ABCD} - PI_{opt} > 0$$

$$\text{Define } \epsilon_2 = \epsilon_3 + \epsilon_4$$

then

$$0 < -\delta + \epsilon_3 + \epsilon_4 < \epsilon_2.$$

It follows that

$$PI_{ABCD} - PI_{opt} < \epsilon_2$$

where  $\epsilon_2 > 0$  is a predetermined small quantity.

Consider the example of problem A. It is now shown that the relaxed problem for problem A has a singular solution.

Consider

$$\dot{x}_0 = \frac{1}{2} x_1^2 + \sqrt{u} \quad x_0(0) = 0 \quad (2.48)$$

$$\dot{x}_1 = x_2 + u \quad x_1(0) = -2 ; x_1(T) = -1 \quad (2.49)$$

$$\dot{x}_2 = -u \quad x_2(0) = 1 ; x_2(T) = 0.5 \quad (2.50)$$

The constraint on  $u$  is  $0 \leq u \leq 1$ .

The solution to (2.48), (2.49) and (2.50) is found by solving the relaxed equations.

$$\dot{x}_0 = \frac{1}{2} x_1^2 + \alpha u_1^{1/2} + (1-\alpha) u_2^{1/2} \quad (2.51)$$

$$\dot{x}_1 = x_2 + \alpha u_1 + (1-\alpha) u_2 \quad x_1(0) = -2 ; x_1(T) = -1 \quad (2.52)$$

$$\dot{x}_2 = -\alpha u_1 - (1-\alpha) u_2 \quad x_2(0) = 1 ; x_2(T) = 0.5 \quad (2.53)$$

with the constraints

$$0 \leq u_1 \leq 1$$

$$0 \leq u_2 \leq 1$$

$$0 \leq \alpha \leq 1$$

It is necessary to consider here only two  $\alpha$ 's instead of four, since one is eliminated because the set of admissible controls  $U$  is connected and the second is eliminated since the control  $u$  appears linearly in both equation (2.49) and (2.50).

The Hamiltonian for (2.51), (2.52) and (2.53) satisfies

$$\begin{aligned} H = \frac{1}{2} x_1^2 + \alpha u_1^{1/2} + (1-\alpha) u_2^{1/2} + \lambda_1 [x_2 + \alpha u_1 + (1-\alpha) u_2] \\ + \lambda_2 [-\alpha u_1 - (1-\alpha) u_2] \end{aligned} \quad (2.54)$$

Observe that

$$\frac{\partial^2 H}{\partial u_1^2} = -1/4 \alpha u_1^{-3/2} \quad (2.55)$$

and

$$\frac{\partial^2 H}{\partial u_2^2} = -1/4 (1-\alpha) u_2^{-3/2} \quad (2.56)$$

It was shown that equations (2.48), (2.49) and (2.50) do not possess an optimal solution. Comparing those equations to equations (2.51), (2.52) and (2.53) it is thus clear that we require  $\alpha \neq 0$  and  $\alpha \neq 1$ .

It follows from (2.55) and (2.56) that the only possible minimum points are  $u_1 = 0, u_1 = 1, u_2 = 0, u_2 = 1$ .

Choose  $u_1 = 1$

$$u_2 = 0$$

Equations (2.51, 2.52, 2.53) become

$$\dot{x}_0 = \frac{1}{2} x_1^2 + \alpha \quad (2.57)$$

$$\dot{x}_1 = x_2 + \alpha \quad (2.58)$$

$$\dot{x}_2 = -\alpha \quad (2.59)$$

Define the Hamiltonian

$$H = \frac{1}{2} x_1^2 + \alpha + \lambda_1 [x_2 + \alpha] + \lambda_2 (-\alpha) \quad (2.60)$$

Then

$$\dot{\lambda}_1 = -x_1 \quad (2.61)$$

and

$$\dot{\lambda}_2 = -\lambda_1 \quad (2.62)$$

Using the notation of [ref. 4], we have

$$I = \frac{1}{2} x_1^2 + \lambda_1 x_2 \quad (2.63)$$

$$F = 1 + \lambda_1 - \lambda_2 \quad (2.64)$$

We now seek a singular solution of the problem defined by equations (2.57), (2.58) and (2.59) together with the initial and final conditions.

$$F = 0 \rightarrow \lambda_1 = \lambda_2 - 1 \quad (2.65)$$

$$\dot{F} = 0 \rightarrow -x_1 + \lambda_1 = 0 \rightarrow x_1 = \lambda_1 \quad (2.66)$$

$$I = 0 \rightarrow \lambda_1 x_2 = -\frac{1}{2} x_1^2 \quad (2.67)$$

$$\dot{I} = 0 \rightarrow x_1(x_2 + \alpha) - x_1 x_2 - \lambda_1 \alpha = 0$$

$$x_1 = \lambda_1 \quad (2.68)$$

Substitute (2.68) into (2.67)

$$x_1 x_2 + \frac{1}{2} x_1^2 = 0$$

$$x_1 [x_1 + 2x_2] = 0 \quad (2.69)$$

(2.69) provides the singular segments.

From (2.69), (2.58) and (2.59) one gets

$$\alpha^* = x_2 \quad (2.70)$$

Using  $u_1 = 1$ ,  $u_2 = 0$  and  $\alpha^* = x_2$  one can construct a sub-optimal control for problem A.

## CHAPTER THREE

### SENSITIVITY IN CONTROL THEORY

#### 3.1 Introduction

Sensitivity considerations play an important role in the design of automatic control systems. One of the major reasons for using feedback for systems is to reduce sensitivity to plant parameter variations and external disturbances. In this chapter a brief review of sensitivity consideration in classical control theory is given. Some work is reported on zero sensitivity systems, and zero sensitivity of relay systems is discussed when the system is in the chattering mode.

The classical definitions of sensitivity are extended to the field of optimal control. Since the formulation of the deterministic optimal control problem does not normally take sensitivity into account, it is necessary to consider sensitivity separately. Finally, an example solved by Holtzman and Horing is discussed.

#### 3.2 Sensitivity Considerations in Classical Control Theory

One of the main reasons for employing feedback in automatic control systems is that it has the ability to reduce sensitivity of the performance of the system to plant parameter variations and to external disturbances. The study of sensitivity analysis is important because system parameter values generally differ slightly from the computed

ones. Mathematical models of systems are idealized representations of the real systems. Also most systems contain components that change their values with time because of aging and wearing. A survey of research on sensitivity of automatic control systems is given by Kokotovic and Rutman [Ref. 16]. The basic concepts of sensitivity appeared in the fundamental work of Bode [Ref. 17] which was the beginning of the modern theory of feedback systems. One would have expected that automatic control theory would include the study of control system sensitivity. However with a few exceptions like the books by Truxal [Ref. 18] and Horowitz [Ref. 19], the question of sensitivity did not even find a place in the texts on automatic control theory.

Truxal's definition [Ref. 18] of sensitivity is summarized below. The sensitivity of an overall gain or transmittance  $T$  with respect to a given parameter  $k$  is defined by the equation

$$S_k^T = \frac{d \ln T}{d \ln k} \quad (3.1)$$

Equation (3.1) can be written as

$$S_k^T = \frac{dT/T}{dk/k} \quad (3.2)$$

The sensitivity of  $T$  with respect to  $k$  is the percentage change in  $T$ , divided by that percentage change in  $k$  which caused the change in  $T$ , with all changes considered infinitesimally small. Since only the first derivatives are involved in the definition of sensitivity, the sensitivity is a measure of system characteristics only for very small changes in the parameter. Specifically, the fact that the sensitivity is small does not guarantee that higher derivatives are also small.

### 3.3 Sensitivity of Variable Structure Systems

Variable structure systems are automatic control systems in which the structure and the parameters of the controller change in correspondence with a chosen logical law as functions of the state of the system.

Variable Structure Systems were investigated by Bermant, Emelyanov and Taran [Ref. 20]. The properties of feedback systems in the sliding state were examined.

Consider the system described by the following block diagram (Figure 3.1), where  $W$  is the plant to be controlled and  $W_g$  is an equalizer.

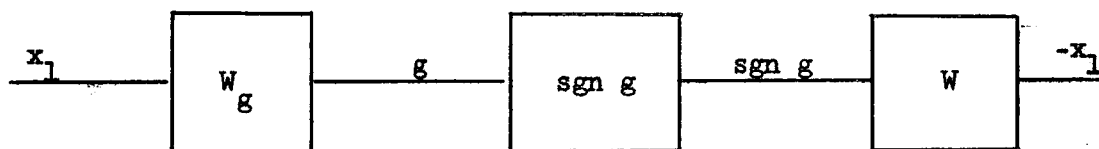


Figure 3.1 Relay Feedback System

Assume that

$$g = \sum_{i=0}^{n-1} c_i \frac{d^i x_1}{dt^i} \quad (3.3)$$

where the  $c_i$  are constants. Also assume that sliding as defined below does indeed occur. Sliding is defined as the situation when  $g$  oscillates around the value zero with infinitely large frequency and infinitesimally small amplitude. In this case it can be assumed [Ref. 20] that

$$\sum_{i=0}^{n-1} c_i \frac{d^i x_1}{dt^i} = 0 \quad (3.4)$$

Assume that the differential equation (3.4) is stable, and that

$$x(t) = \text{col} \left( x_1, \frac{dx_1}{dt}, \dots, \frac{d^{n-2} x_1}{dt^{n-2}} \right)$$

and

$$x^*(t) = \text{col} \left( x_1^*, \frac{dx_1^*}{dt}, \dots, \frac{d^{n-2} x_1^*}{dt^{n-2}} \right)$$

Thus, if  $x_1^*(t)$  is the solution to equation (3.4), an initial condition  $x(0)$  will be transferred to the origin following  $x^*(t)$ , independent of  $W$ . As long as sliding exists, the motion of  $x(t)$  will follow approximately that of  $x^*(t)$ , with no dependence on the plant  $W$ . It is thus concluded that the transient response of the system, when it is in sliding, is insensitive to plant parameter variation of the plant  $W$ .

### 3.4 Definitions of Sensitivity in Optimal Control

Classical sensitivity has recently been employed by Dorato [Ref. 9] who discussed the sensitivity of the performance index in the optimal control problem with respect to plant parameter variations. A method was outlined for computing the performance index sensitivity functions. A general system was considered.

$$\dot{x} = f(x, u, w) \quad (3.5)$$

$$\text{where } x = \text{col} (x_1, x_2, \dots, x_n) \quad (3.6)$$

$$u = \text{col} (u_1, u_2, \dots, u_m) \quad (3.7)$$

$$\text{and } w = \text{col} (w_1, w_2, \dots, w_p) \quad (3.8)$$

where  $u_i$  are controls and  $w_j$  are parameters of the system.

A performance index to be minimized was

$$S = \int_{t_0}^T F(x, u) dt \quad (3.9)$$

It was assumed that an optimal feedback solution exists in the form

$$u^0(t) = \psi(x, w_0, t) \quad (3.10)$$

where  $w_0$  is the nominal value for the plant parameters. The closed loop system dynamics were described by

$$\dot{x} = f[x, \psi(x, w_0, t), w] \quad (3.11)$$

with a corresponding performance index value  $S(w_0, w)$ . Variations in  $S$  due to plant parameter variations were represented by

$$\Delta S = S(w_0, w) - S(w_0, w_0) \quad (3.12)$$

### 3.5 Discussion of an Example Given by Holtzman and Horing [Ref. 14]

$$\text{Given } \dot{x}_0 = u^2 \quad x_0(0) = 0 \quad (3.13)$$

$$\dot{x}_1 = ax_1 + bu \quad x_1(0) = x_0 ; x_1(T) = x_T \quad (3.14)$$

Find an optimal control  $u^*(t)$ ,  $0 \leq t \leq T$ , that will minimize  $x_0(T)$ , where  $T$  is fixed

$$x_0(T) = \int_0^T u^2 dt \quad (3.15)$$

Define sensitivity for this problem as

$$S = \frac{\delta x_1(T)}{\delta a} \approx \frac{dx_1(T)}{da} \quad (3.16)$$

The approximation of the sensitivity by the derivative  $\frac{dx_1(T)}{da}$  is valid for small values of  $\delta a$  when higher order terms can be neglected.

An equation is derived to measure the sensitivity. From (3.14)

$$x_1(t) = \int_0^t \dot{x}_1 dt + x_0 = \int_0^t (a x_1 + bu) dt + x_0 \quad (3.17)$$

Take the derivative of (3.17) with respect to  $a$ .

$$\frac{dx_1(t)}{da} = \int_0^t \left\{ x_1 + a \frac{dx_1(t)}{da} \right\} dt \quad (3.18)$$

Take the derivative of (3.18) with respect to  $t$ .

$$\frac{d}{dt} \left[ \frac{dx_1(t)}{da} \right] = x_1 + a \left[ \frac{dx_1(t)}{da} \right] \quad (3.19)$$

Define

$$\frac{dx_1(t)}{da} \triangleq x_2(t)$$

and rewrite equations (3.13) and (3.14) to get a set of three equations

$$\dot{x}_0 = u^2 \quad x_0(0) = 0 \quad (3.20)$$

$$\dot{x}_1 = ax_1 + bu \quad x_1(0) = x_0 ; x_1(T) = x_T \quad (3.21)$$

$$\dot{x}_2 = ax_2 + x_1 \quad x_2(0) = 0 ; x_2(T) = S \quad (3.22)$$

$x_2(T) = S$  in (3.22) puts a constraint on the sensitivity. The optimal solution will have sensitivity equal to  $S$ .

The optimization problem given by (3.20), (3.21) and (3.22) was solved, [Ref. 14]. It was assumed that  $b = 1.0$  and  $T = 1.0$ .

In the formulation given above,  $S$  could be chosen to be any real number. For  $S = 0$  one would get an optimal control and trajectory with zero sensitivity with respect to  $a$ . However this conclusion is misleading since it was shown in the paper that for  $S = 0$ , different performance indices result for different values of the parameter  $a$ , which is a contradiction to zero sensitivity, as discussed below.

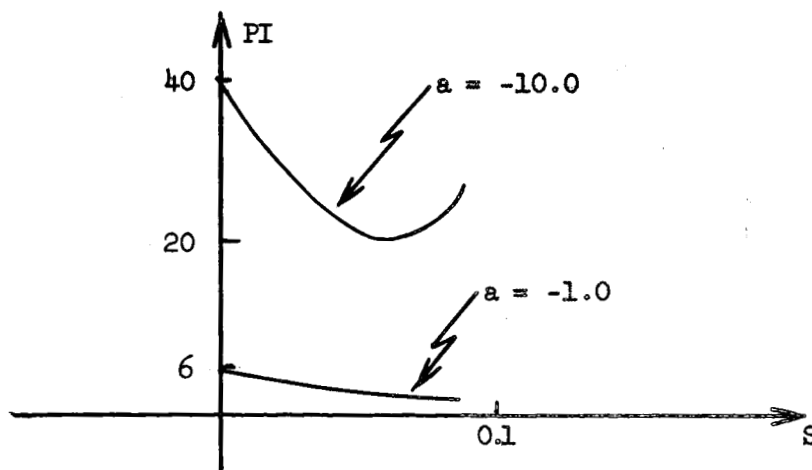


Figure 3.2 PI as a Function of Sensitivity

As seen from Figure 3.2, for zero sensitivity case  $S = 0$  a performance index of 40 is obtained for  $a = -10$  while a value of  $PI = 6$  is obtained for  $a = -1.0$ . If indeed  $S = 0$ , one could use the control computed for  $a = -1.0$  for the case  $a = -10$ , and thus achieve a lower performance index. The results are misleading because of the fact that sensitivity for a finite perturbation  $\frac{\delta x_1(T)}{\delta a}$  was approximated by the derivative  $\frac{dx_1(T)}{da}$  neglecting higher order terms.

## CHAPTER FOUR

## EXAMPLES OF MINIMUM SENSITIVITY OPTIMAL CONTROL

Two examples of minimum sensitivity optimal control are presented here. In the first example a sub-optimal open loop chattering control is made a function of the states and thus becomes a feedback control. In this configuration the system will follow approximately the optimal trajectory even when there are small perturbations in the initial states. In the second example relaxed variational techniques are applied to the construction of a sub-optimal control to minimize sensitivity of a trajectory to perturbations in initial conditions.

Example 4.1. Gurman's Problem.

The following example was considered by Gurman [ref. 8]. In his work, the sliding state of the angle of attack of an airplane is considered. It is desired to transfer the airplane from an initial state to a final state in minimum time. The state of the system consists of the height, velocity and angle between the longitudinal axis of the airplane and the horizontal plane. It is shown that in order to satisfy a state variable constraint

$$\frac{1}{2} \rho v^2 \leq q_{\max}$$

where  $\rho$  is the density of the atmosphere and  $v$  is the velocity of the airplane, a sliding regime results. The sliding control is a sub-optimal control for the problem that does not possess an optimal solution

In this example it is shown that the control when it is chattering, can be expressed as a function of the state of the system and of the optimal trajectory of the relaxed problem. It is shown how to choose a switching function such that the control chatters while leading the system along the trajectory that satisfies the given state variable constraint.

The equations of the motion of the system are [ref. 8]:

$$\begin{aligned} \dot{h} &= v \sin \theta & h(0) &= h_0 & h(T) &= h_T \\ \dot{v} &= -X - f \sin \theta & v(0) &= v_0 & v(T) &= v_T \\ \dot{\theta} &= \frac{1}{v} \left[ Y + \left( \frac{v^2}{R_3} - f \right) \cos \theta \right] & \theta(0) &= \theta_0 & \theta(T) &= \theta_T \end{aligned} \quad (4.1)$$

where

$C_L$  : control  
 $X = X[h, v, C_D]$ : drag  
 $C_D = C_D(C_L)$  : drag coefficient  
 $Y = Y[h, v, C_L]$ : lift  
 $h$  : height  
 $v$  : velocity  
 $\theta$  : angle between horizontal plane and the longitudinal axis of the airplane  
 $f$  : gravity constant  
 $R_3$  : radius of the earth

The control  $C_L$  is constrained

$$C_{L_1} \leq C_L \leq C_{L_2}$$

where  $C_{L_1}$  and  $C_{L_2}$  are given constants. The trajectory must satisfy a constraint

$$\frac{1}{2} \rho(h) v^2 \leq q_{\max}$$

where  $\rho(h)$  is the density of the atmosphere. It is shown [ref. 8] that for some given initial and final conditions the limiting optimal trajectory includes the path  $\frac{1}{2} \rho v^2 = q_{\max}$ .

This example is concerned only with that part of the optimal trajectory that satisfies the state variable constraint  $\frac{1}{2} \rho v^2 = q_{\max}$ . Choose initial and final conditions that belong to this part of the optimal trajectory that satisfies  $\frac{1}{2} \rho v^2 = q_{\max}$ . Since every part of an optimal trajectory is also optimal, it follows that the new problem with the new set of initial and final conditions has an optimal solution that satisfies  $\frac{1}{2} \rho v^2 = q_{\max}$ . It is assumed here that the given initial conditions satisfy the constraint  $\frac{1}{2} \rho v^2 = q_{\max}$  and that it is desired to follow a trajectory that satisfies the same constraint. A chattering feedback control will be found for this problem.

It can be shown [ref. 21] that

$$X = \frac{1}{2} \rho v^2 S C_D \frac{1}{m}$$

$$Y = \frac{1}{2} \rho v^2 S C_L \frac{1}{m}$$

where  $S$  is the area of the wings and  $m$  is the mass of the airplane.

Substituting  $X$  and  $Y$  into equation (4.1) one obtains

$$\begin{aligned} \dot{h} &= v \sin \theta \\ \dot{v} &= -\frac{1}{2} \rho v^2 S C_D \frac{1}{m} - f \sin \theta \\ \dot{\theta} &= \frac{1}{v} \left[ \frac{1}{2} \rho v^2 S C_L \frac{1}{m} + \left( \frac{v^2}{R_3} - f \right) \cos \theta \right] \end{aligned} \quad (4.2)$$

From Perkins and Hage [ref. 21] pp. 94, 481

$$C_{D_f} = C_{D_f} + K C_L^2 = 0.025 + 0.06 C_L^2$$

and

$$\rho/\rho_0 = [1 - 22.6 \cdot 10^{-6} \cdot h]^{4.3}$$

where  $\rho_0$  is the density of the atmosphere at sea level. The last expression is approximate and holds only for values of  $h$  that satisfy:

$$1 - 22.6 \cdot 10^{-6} h > 0$$

Assume that the lift is approximately 10 f when  $C_L = 1$

$$\frac{1}{2} \rho_0 v_0^2 S \frac{1}{m} = 100 \text{ m/sec}^2$$

$$-1 \leq C_L \leq +1$$

$$v_0 = 100 \text{ m/sec}$$

and neglecting the term  $v^2/R_3$  compared to  $f$ , (4.2) becomes:

$$\dot{h} = v \sin \theta$$

$$\begin{aligned} \dot{v} = & -2.5 \cdot 10^{-4} (1 - 22.6 \cdot 10^{-6} h)^{4.3} v^2 - \\ & - 6.10^{-4} (1 - 22.6 \cdot 10^{-6} h)^{4.3} v^2 C_L^2 - 9.8 \sin \theta \end{aligned} \quad (4.3)$$

$$\dot{\theta} = \frac{1}{v} [0.01 (1 - 22.6 \cdot 10^{-6} h)^{4.3} v^2 C_L - 9.8 \cos \theta]$$

Assume the given constraint to be

$$\frac{1}{\rho_0} \rho v^2 \leq 3322 \text{ m}^2/\text{sec}^2 \quad (4.4)$$

and that the initial conditions (4.5) satisfy equation (4.4a)

$$\frac{1}{\rho_0} \rho v^2 = 3322 \text{ m}^2/\text{sec}^2 \quad (4.4a)$$

$$h(0) = 10^4 \text{ meters}$$

$$v(0) = 100 \text{ m/sec} \quad (4.5)$$

$$\theta(0) = 0.0 \text{ rad.}$$

It is desired to construct a feedback control that will transfer system (4.3) from its initial state (4.5) along the trajectory satisfying

$$\frac{1}{\rho_0} \rho v^2 = 3322 \text{ m}^2/\text{sec}^2$$

A switching function  $g(h, v, \theta)$  has to be chosen such that

$$g \frac{dg}{dt} < 0 \quad (4.6)$$

If (4.6) is satisfied,  $g$  will tend to become smaller when it is positive and larger when it is negative, and will thus always tend toward the value  $g = 0$ .  $g$  should be chosen such that  $g = 0$  satisfies the constraint (4.4a) and then the system will follow the trajectory resulting from the constraint. This is of course true provided other solutions of  $g = 0$  are not solutions of (4.3).

The following switching function was tried.

$$g = v^2 (1 - 22.6 \cdot 10^{-6} \cdot h)^{4.3} - 3322 - K(\theta - \theta_N) \quad (4.7)$$

where  $\theta_N$  is the angle between the desired trajectory, satisfying the constraint (4.4a), and the horizontal plane.

$\theta_N$  is computed as follows. Along the optimal trajectory the following constraint holds.

$$v^2 (1 - ah)^{4.3} - b = 0 \quad (4.8)$$

where  $a = 22.6 \cdot 10^{-6}$

$$b = 3322$$

It follows from (4.8) that

$$v = \sqrt{b} (1 - ah)^{-4.3/2} \quad (4.9)$$

Taking the derivative of (4.9) with respect to  $h$  yields

$$\frac{dv}{dh} = \sqrt{b} \frac{4.3}{2} a (1 - ah)^{-4.3/2} - 1 \quad (4.10)$$

$$\frac{dv}{dh} = 2.15 \sqrt{b} a (1 - ah)^{-3.15} \quad (4.11)$$

But from equation (4.3)

$$\begin{aligned} \frac{dv}{dh} = \frac{1}{v \sin \theta} \{ & -2.5 \cdot 10^{-4} (1 - ah)^{4.3} v^2 - \\ & - 6 \cdot 10^{-4} (1 - ah)^{4.3} v^2 C_L^2 - 9.8 \sin \theta \} \end{aligned} \quad (4.12)$$

Substituting (4.8) into (4.12) yields for  $C_L^2 = 1$ , (it was shown in [ref. 8] that the base controls for the relaxed problem are  $C_L = 1$  and  $C_L = -1$ , thus  $C_L^2 = 1$ )

$$\begin{aligned}\frac{dv}{dh} &= \frac{1}{v \sin \theta} \{-2.5 \cdot 10^{-4} b - 6 \cdot 10^{-4} b - 9.8 \sin \theta\} \\ \frac{dv}{dh} &= \frac{1}{v \sin \theta} \{-2.8237 - 9.8 \sin \theta\}\end{aligned}\quad (4.13)$$

Substitute (4.11) into (4.13) to obtain

$$\begin{aligned}\sin \theta [v 2.15 \sqrt{b} a (1 - ah)^{-3.15} + 9.8] &= -2.8237 \\ \sin \theta &= \frac{-2.8237}{2.15 v \sqrt{b} a (1 - ah)^{-3.15} + 9.8}\end{aligned}\quad (4.14)$$

By definition  $\theta$  in (4.14) is  $\theta_N$

$$\sin \theta_N = \frac{-2.8237}{2.15 \sqrt{b} a v (1 - ah)^{-3.15} + 9.8}\quad (4.15)$$

The constant  $K$  in (4.7) is chosen large enough to satisfy

$$\text{sgn} \left( \frac{dg}{dt} \right) = - \text{sgn} (g)\quad (4.16)$$

Assume the control is

$$C_L = \text{sgn} (g)\quad (4.17)$$

then

$$\frac{dg}{dt} = 2v (1 - ah)^{4.3} \dot{v} - 4.3a (1 - ah)^{3.3} v^2 \dot{h} - K\dot{\theta}\quad (4.18)$$

Substitute (4.3) into (4.18)

$$\begin{aligned}\frac{dg}{dt} &= 2v (1 - ah)^{4.3} \{-2.5 \cdot 10^{-4} (1 - ah)^{4.3} v^2 - \\ &\quad - 6 \cdot 10^{-4} (1 - ah)^{4.3} v^2 - 9.8 \sin \theta\} - \\ &\quad - 4.3 a v^3 (1 - ah)^{3.3} \sin \theta - \\ &\quad - K \left\{ \frac{1}{v} [0.01 (1 - ah)^{4.3} v^2 \text{sgn } g - 9.8 \cos \theta] \right\}\end{aligned}\quad (4.19)$$

when

$$0.01 (1 - ah)^{4.3} v^2 > |9.8 \cos \theta|\quad (4.20)$$

K can be chosen large enough to satisfy the condition (4.16).

In our example  $K = 100 \text{ m}^2/\text{sec}^2$  was used and the result was a chattering control,  $C_L$ , between -1 and +1 leading the state of the system close to the state variable constraint (4.4a). The maximum deviation of  $\frac{1}{\rho_0} \rho v^2$  from b was 0.15%. The control and trajectory are given in table 4.1.

When another switching function g

$$g = v^2(1 - ah)^{4.3} - b \quad (4.21)$$

was used, the control did not chatter and the trajectory resulted in a deviation of 4.0% for  $\frac{1}{\rho_0} \rho v^2$  from the desired value of b. The control and trajectory are given in table 4.2.

It remains to be shown that the only possible solution to  $g = 0$

$$g = v^2(1 - ah)^{4.3} - b - K(\theta - \theta_N) = 0 \quad (4.22)$$

is

$$v^2(1 - ah)^{4.3} = b \quad (4.23)$$

and

$$\theta = \theta_N \quad (4.24)$$

From (4.22)

$$\theta = \frac{1}{K} [v^2(1 - ah)^{4.3} - b] + \theta_N \quad (4.25)$$

or

$$\begin{aligned} \dot{\theta} &= \frac{1}{K} [2v \dot{v} (1 - ah)^{4.3}] + \frac{1}{K} [-4.3 v^2 a (1 - ah)^{3.3} \dot{h}] \\ \dot{\theta} &= \frac{1}{K} [2v (1 - ah)^{4.3} \{-2.5 \cdot 10^{-4} (1 - ah)^{4.3} v^2 - \\ &\quad - 6 \cdot 10^{-4} (1 - ah)^{4.3} v^2 C_L^2 - 9.8 \sin \theta\}] + \\ &\quad + \frac{1}{K} [-4.3 a v^3 (1 - ah)^{3.3} \sin \theta] \end{aligned} \quad (4.26)$$

Table 4.1 Chattering Control and Trajectory for Example  
4.1 when  $g = v^2(1-ah)^{4.5} - b - 100(\theta - \theta_N)$

t	$h-h_0$	v	$\theta$	g	$C_L$	$\frac{1}{\rho_0} \rho v^2 - 3322$
0.0	0	100.0	0.0	0.0	1.0	0.0
0.1	-2.6	99.97	-0.25	1.4	1.0	1.43
0.2	-5.3	99.95	-0.29	-1.8	-1.0	0.63
0.3	-8.0	99.94	-0.27	2.3	1.0	0.46
0.4	-11.0	99.94	-0.25	0.4	1.0	0.92
0.5	-13.8	99.93	-0.30	-0.5	-1.0	2.17
0.6	-16.2	99.89	-0.26	5.5	1.0	2.82
0.7	-20.0	99.93	-0.32	-0.5	-1.0	0.52
0.8	-22.0	99.88	-0.28	9.5	1.0	4.88
0.9	-25.0	99.90	-0.26	4.0	1.0	3.22
1.0	-27.3	99.85	-0.24	3.9	1.0	5.18
1.1	-30.1	99.84	-0.29	-0.8	-1.0	2.93
1.2	-32.4	99.78	-0.25	5.2	1.0	3.47
1.3	-35.6	99.82	-0.31	-1.6	-1.0	0.48
1.4	-38.0	99.77	-0.27	7.7	1.0	4.18
1.5	-40.9	99.77	-0.25	1.6	1.0	1.84

Table 4.2 Feedback Control and Trajectory for Example 4.1 when  $g = v^2(1-ah)^{4.5} - b$

t	$h-h_0$	v	$\theta$	$C_L$	g
0.0	0	100.0	0.0	1.0	0.0
0.3	-0.3	99.19	-0.06	-1.0	-37.
0.6	-4.1	98.74	-0.19	-1.0	-74.
0.9	-11.5	98.65	-0.32	-1.0	-85.
1.2	-22.6	98.91	-0.44	-1.0	-72.
1.5	-37.0	99.50	-0.57	-1.0	-34.
1.8	-53.5	100.27	-0.55	1.0	22.
2.1	-72.8	100.99	-0.48	1.0	73.
2.4	-81.2	101.22	-0.40	1.0	109.
2.7	-92.0	101.39	-0.33	1.0	130.
3.0	-100.6	101.34	-0.25	1.0	135.
3.3	-107.0	101.09	-0.18	1.0	126.
3.6	-111.3	100.63	-0.10	1.0	101.
3.9	-113.4	99.96	-0.03	1.0	62.
4.2	-113.3	99.10	0.04	1.0	9.
4.5	-114.3	98.37	-0.10	-1.0	-12.

Clearly, equation (4.26) differs in form from equation (4.3). It is therefore expected that the solution of (4.26) together with the first two equations of (4.3) are not solutions of (4.3).

#### Example 4.2

In the following example the theory of Relaxed Variational Problems is applied to a problem of minimizing sensitivity of an optimal trajectory to variations in initial conditions. It is demonstrated that relaxing the end conditions allows for a sub-optimal control that results in a lower performance index than the optimal performance index.

Given

$$\dot{x}_0 = 100 x_2^2 + 0.25 u^4 \quad x_0(0) = 0 \quad (4.27)$$

$$\dot{x}_1 = (u - x_1)^3 \quad x_1(0) = 0 \quad (4.28)$$

$$\dot{x}_2 = -3(u - x_1)^2 x_2 \quad x_2(0) = 1 \quad (4.29)$$

$$\dot{x}_3 = x_1^2 \quad x_3(0) = 0 ; x_3(0.1) = 0 \quad (4.30)$$

It is desired to find a control  $u^*(t)$  such that minimizes  $x_0(T)$ , where  $T = 0.1$  is the terminal time.

Equation (4.28) is the plant equation. Equation (4.30) forces a constraint on the trajectory.

$$x_1(t) \equiv 0 \quad (4.31)$$

Equation (4.29) is a linearized perturbation equation for equation (4.28).

$$\delta \dot{x}_1 = \frac{\partial[(u-x_1)^3]}{\partial x_1} \delta x_1 \quad (4.32)$$

$$\delta \dot{x}_1 = -3(u - x_1)^2 \delta x_1 \quad (4.33)$$

Defining  $x_2 \triangleq \delta x_1$  reduces equation (4.33) to equation (4.29).

It is assumed that there might be a perturbation of magnitude 1 in

the initial condition and thus

$$x_2(0) = 1.0$$

The optimal control problem thus formulated will be called the original problem. The original problem has only one admissible control,  $u^*(t) \equiv 0$ , that transfers the initial state to the final state and is therefore also the optimal control for the problem.

From equation (4.30) it is clear that  $x_1 \equiv 0$  and from (4.28)

$$u^*(t) = 0 \quad (4.34)$$

The optimal performance index considering  $\dot{x}_2 = 0$  from (4.29) is

$$x_0(0.1) = \int_0^{0.1} 100 \, dt = 10 \quad (4.35)$$

The relaxed control problem will be defined and solved below. It will be shown that the performance index for the relaxed problem is considerably lower than the one for the original problem. A sub-optimal control with a finite number of switchings will be constructed for the original problem and it will be shown that the performance index thus obtained is still considerably lower than the optimal. This will be done by sacrificing the terminal constraints.

Define the relaxed equations for (4.27), (4.28), (4.29) and (4.30).

$$\dot{x}_0 = 100 x_2^2 + 0.25 [\alpha_1 u_1^4 + \alpha_2 u_2^4 + (1-\alpha_1-\alpha_2) u_3^4] \quad x_0(0)=0$$

$$\dot{x}_1 = \alpha_1 (u_1 - x_1)^3 + \alpha_2 (u_2 - x_1)^3 + (1-\alpha_1-\alpha_2) (u_3 - x_1)^3 \quad x_1(0)=0$$

$$\dot{x}_2 = -3x_2 [\alpha_1 (u_1 - x_1)^2 + \alpha_2 (u_2 - x_1)^2 + (1-\alpha_1-\alpha_2) (u_3 - x_1)^2] \quad x_2(0)=1$$

$$\dot{x}_3 = x_1^2 \quad x_3(0)=0 \quad x_3(0.1)=0$$

It is desired to minimize  $x_0(T)$  where  $T = 0.1$  is the terminal time.

From the fourth equation and from initial and final values it is

clear that  $x_1 \equiv 0$  and that  $\dot{x}_1 = 0$ . Applying the Minimum Principle to this problem we find that the Hamiltonian satisfies:

$$H = 100 x_2^2 + 0.25 [\alpha_1 u_1^4 + \alpha_2 u_2^4 + (1-\alpha_1-\alpha_2)u_3^4] \\ - 3\lambda_2 x_2 [\alpha_1 u_1^2 + \alpha_2 u_2^2 + (1-\alpha_1-\alpha_2)u_3^2]$$

If any one of the  $\alpha$ 's is zero,  $\alpha_1 = 0$ ,  $\alpha_2 = 0$  or  $(1-\alpha_1-\alpha_2) = 0$ , then the problem reduces to the problem discussed below equation (4.36) with only two  $\alpha$ 's. Assume none of the  $\alpha$ 's is zero, then applying the necessary conditions the following must hold

$$\frac{\partial H}{\partial u_1} = \alpha_1(u_1^3 - 6\lambda_2 x_2 u_1) = 0$$

$$\frac{\partial H}{\partial u_2} = \alpha_2(u_2^3 - 6\lambda_2 x_2 u_2) = 0$$

$$\frac{\partial H}{\partial u_3} = (1-\alpha_1-\alpha_2)(u_3^3 - 6\lambda_2 x_2 u_3) = 0$$

Since it was assumed  $\alpha_1 \neq 0$ ,  $\alpha_2 \neq 0$ ,  $1 - \alpha_1 - \alpha_2 \neq 0$ , then the following must hold

$$u_1^3 - 6\lambda_2 x_2 u_1 = 0$$

$$u_2^3 - 6\lambda_2 x_2 u_2 = 0$$

$$u_3^3 - 6\lambda_2 x_2 u_3 = 0$$

This results in

$$u_1 = 0 \text{ or } u_1 = \pm \sqrt{6\lambda_2 x_2} \text{ if } \lambda_2 x_2 > 0$$

$$u_2 = 0 \text{ or } u_2 = \pm \sqrt{6\lambda_2 x_2} \text{ if } \lambda_2 x_2 > 0$$

$$u_3 = 0 \text{ or } u_3 = \pm \sqrt{6\lambda_2 x_2} \text{ if } \lambda_2 x_2 > 0$$

Also

$$\frac{\partial^2 H}{\partial u_1^2} = \alpha_1 (3u_1^2 - 6\lambda_2 x_2)$$

If  $\lambda_2 x_2 < 0$ ,  $u_1 = u_2 = u_3 = 0$  is the only solution. Assume  $\lambda_2 x_2 > 0$ , then

$$\frac{\partial^2 H}{\partial u_1^2} = -6\alpha\lambda_2 x_2 < 0$$

for  $u_1 = 0$  and

$$\frac{\partial^2 H}{\partial u_1^2} = 12\alpha\lambda_2 x_2 > 0$$

for  $u_1 = \sqrt{6\lambda_2 x_2}$  and for  $u_1 = -\sqrt{6\lambda_2 x_2}$ .

The same can be done for  $\frac{\partial^2 H}{\partial u_2^2}$  and for  $\frac{\partial^2 H}{\partial u_3^2}$  with the result

$$u_1^* = \pm \sqrt{6\lambda_2 x_2}$$

$$u_2^* = \pm \sqrt{6\lambda_2 x_2}$$

and  $u_3^* = \pm \sqrt{6\lambda_2 x_2}$

It is clear that at least two out of the three controls  $u_1$ ,  $u_2$ , and  $u_3$  have to be identical, therefore it is sufficient to use only two controls  $u_1$  and  $u_2$  and two  $\alpha$ 's,  $\alpha$  and  $(1-\alpha)$ . This is shown in equation (4.36).

$$\begin{aligned} \dot{x}_0 &= 100 x_2^2 + 0.25 [\alpha u_1^4 + (1-\alpha) u_2^4] & x_0(0) &= 0 \\ \dot{x}_1 &= \alpha(u_1 - x_1)^3 + (1-\alpha)(u_2 - x_1)^3 & x_1(0) &= 0 \\ \dot{x}_2 &= -3 x_2 [\alpha(u_1 - x_1)^2 + (1-\alpha)(u_2 - x_1)^2] & x_2(0) &= 1 \\ \dot{x}_3 &= x_1^2 & x_3(0) &= 0 \quad x_3(0.1) = 0 \end{aligned} \quad (4.36)$$

It is desired to minimize  $x_0(T)$  where  $T = 0.1$  is the terminal time.

Applying the Minimum Principle to this problem we find that the

Hamiltonian satisfies

$$\begin{aligned}
 H = & 100 x_2^2 + 0.25 [\alpha u_1^4 + (1-\alpha) u_2^4] + \\
 & + \lambda_1 [\alpha(u_1 - x_1)^3 + (1-\alpha)(u_2 - x_1)^3] + \\
 & + \lambda_2 \{-3x_2[\alpha(u_1 - x_1)^2 + (1-\alpha)(u_2 - x_1)^2]\} \\
 & + \lambda_3 x_1^2
 \end{aligned} \tag{4.37}$$

From the fourth equation of (4.36) it follows that

$$x_1 \equiv 0 \tag{4.38}$$

thus

$$\dot{x}_1 = 0 \tag{4.39}$$

and from (4.38) and the second equation of (4.36) it follows that

$$\alpha u_1^3 + (1-\alpha) u_2^3 = 0 \tag{4.40}$$

Using (4.38) and (4.40), the Hamiltonian (4.37) can be rewritten.

$$\begin{aligned}
 H = & 100 x_2^2 + 0.25 [\alpha u_1^4 + (1-\alpha) u_2^4] + \\
 & + \lambda_2 \{-3x_2[\alpha u_1^2 + (1-\alpha) u_2^2]\}
 \end{aligned} \tag{4.41}$$

Applying the necessary conditions the following must hold.

$$\frac{\partial H}{\partial u_1} = \alpha(u_1^3 - 6\lambda_2 x_2 u_1) = 0 \tag{4.42}$$

A similar expression can be written for the partial derivative of H with respect to  $u_2$ .

Two cases are to be considered for (4.42).

Case I:  $\alpha \equiv 0$

The case when  $\alpha = 0$  is of no interest, since in this case the relaxed equations (4.36) reduce to the original equations (4.27), (4.28), (4.29) and (4.30), and that problem was already solved with

$$u_2^* = u^* = 0$$

Case II:  $\alpha \neq 0$

and

$$u_1^3 - 6\lambda_2 x_2 u_1 = 0 \quad (4.43)$$

The solution to (4.43) is

$$u_1 = 0 \quad (4.44)$$

$$u_1 = \sqrt{6\lambda_2 x_2} \quad \text{when } \lambda_2 x_2 > 0 \quad (4.45)$$

$$u_1 = -\sqrt{6\lambda_2 x_2} \quad \text{when } \lambda_2 x_2 > 0 \quad (4.46)$$

Assume  $\lambda_2 x_2 > 0$ , which turns out to be the case in the numerical solution, and compute the second partial derivative of  $H$  with respect to  $u_1$ .

$$\frac{\partial^2 H}{\partial u_1^2} = \alpha(3u_1^2 - 6\lambda_2 x_2) \quad (4.47)$$

$$\frac{\partial^2 H}{\partial u_1^2} = -6\alpha \lambda_2 x_2 < 0 \quad (4.48)$$

for  $u_1 = 0$

and

$$\frac{\partial^2 H}{\partial u_1^2} = 12\alpha \lambda_2 x_2 > 0 \quad (4.49)$$

for  $u_1 = \sqrt{6\lambda_2 x_2}$  and for  $u_1 = -\sqrt{6\lambda_2 x_2}$ .

Thus if  $\lambda_2 x_2 > 0$ ,

$$u_1^* = \sqrt{6\lambda_2 x_2} \quad (4.50)$$

and

$$u_1^* = -\sqrt{6\lambda_2 x_2} \quad (4.51)$$

are minimum points for the Hamiltonian, and  $u_1 = 0$  is a maximum point and is therefore discarded here.

If  $\lambda_2 x_2$  were negative,  $u^* = 0$  would have to be used.

The same argument starting with equation (4.42) can be repeated for  $u_2$  with the results

$$u_2^* = \sqrt{6\lambda_2 x_2} \quad (4.52)$$

and

$$u_2^* = -\sqrt{6\lambda_2 x_2} \quad (4.53)$$

#### Choice of Signs of the Control

It is now shown, that if  $u_1^*$  is chosen to be positive

$$u_1^* = \sqrt{6\lambda_2 x_2}$$

then  $u_2^*$  has to be chosen to be negative

$$u_1^* = -\sqrt{6\lambda_2 x_2}$$

This is shown by contradiction. From equation (4.40)

$$\alpha u_1^3 + (1-\alpha) u_2^3 = 0 \quad (4.54)$$

Assume  $u_1^* = u_2^* = \sqrt{6\lambda_2 x_2}$ , then

$$(\sqrt{6\lambda_2 x_2})^3 [\alpha + (1-\alpha)] = 0$$

Since  $\sqrt{6\lambda_2 x_2} \neq 0$ , it follows

$$\alpha + (1-\alpha) = 0$$

which is a contradiction.

#### Computation of $\alpha$

From equation (4.40)

$$\alpha u_1^3 + (1-\alpha) u_2^3 = 0 \quad (4.55)$$

Choose

$$u_1 = \sqrt{6\lambda_2 x_2} \quad (4.56)$$

and

$$u_2 = -\sqrt{6\lambda_2 x_2} \quad (4.57)$$

Substitute (4.56) and (4.57) into (4.55)

$$\alpha [\sqrt{6\lambda_2 x_2}]^3 + (1-\alpha) [-\sqrt{6\lambda_2 x_2}]^3 = 0 \quad (4.58)$$

$$[\sqrt{6\lambda_2 x_2}]^3 [\alpha - (1-\alpha)] = 0$$

Since  $\sqrt{6\lambda_2 x_2} \neq 0$  for  $0 \leq t < T$ , it follows

$$\alpha - 1 + \alpha = 0$$

$$\alpha = 0.5 \quad (4.59)$$

#### The Adjoint Equation

From (4.41) one gets

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -200 x_2 + 3\lambda_2 [\alpha u_1^2 + (1-\alpha) u_2^2] \quad (4.60)$$

Substitute

$$\begin{aligned} u_1^2 &= u_2^2 = 6\lambda_2 x_2 \\ \alpha &= 0.5 \end{aligned} \quad (4.61)$$

to get

$$\dot{\lambda}_2 = -200 x_2 + 18 \lambda_2^2 x_2 \quad (4.62)$$

From the third equation of (4.36) and (4.38)

$$\dot{x}_2 = -3x_2 [\alpha u_1^2 + (1-\alpha) u_2^2] \quad (4.63)$$

Substitute (4.61) into (4.63) to get

$$\dot{x}_2 = -18 x_2^2 \lambda_2 \quad (4.64)$$

Thus the following two point boundary value problem is to be solved.

Equations (4.62) and (4.64) are recopied.

$$\dot{x}_2 = -18 \lambda_2 x_2^2 \quad x_2(0) = 1.0 \quad (4.65)$$

$$\dot{\lambda}_2 = -200 x_2 + 18 \lambda_2^2 x_2 \quad \lambda_2(0.1) = 0.0 \quad (4.66)$$

The final condition for  $\lambda_2$  is obtained from the transversality condition, noting that  $x_2$  at the terminal time is free.

Once equations (4.65) and (4.66) are solved, the performance index for the relaxed problem can be computed from the first equation of (4.36)

$$\dot{x}_0 = 100 x_2^2 + 0.25 [\alpha u_1^4 + (1-\alpha) u_2^4] \quad (4.67)$$

Substituting (4.61) into (4.67) results in

$$\dot{x}_0 = 100 x_2^2 + 9.0 \lambda_2^2 x_2^2 \quad (4.68)$$

#### Numerical Solution of the Two Point Boundary Value Problem

A Gradient Technique was employed for the solution of the two point boundary value problem, equations (4.65) and (4.66).

Two adjoint equations are constructed for (4.65) and (4.66)

$$\delta \dot{x}_2 = \frac{\partial}{\partial x_2} [-18 \lambda_2 x_2^2] \delta x_2 + \frac{\partial}{\partial \lambda_2} [-18 \lambda_2 x_2^2] \delta \lambda_2 \quad (4.69)$$

$$\begin{aligned} \delta \dot{\lambda}_2 = & \frac{\partial}{\partial x_2} [-200 x_2 + 18 \lambda_2^2 x_2] \delta x_2 + \\ & + \frac{\partial}{\partial \lambda_2} [-200 x_2 + 18 \lambda_2^2 x_2] \delta \lambda_2 \end{aligned} \quad (4.70)$$

(4.69) and (4.70) become

$$\delta \dot{x}_2 = -36 \lambda_2 x_2 \cdot \delta x_2 - 18 x_2^2 \cdot \delta \lambda_2 \quad (4.71)$$

$$\delta \dot{\lambda}_2 = (-200 + 18 \lambda_2^2) \cdot \delta x_2 + 36 \lambda_2 x_2 \cdot \delta \lambda_2 \quad (4.72)$$

Define  $y_1 = \delta x_2$

$y_2 = \delta \lambda_2$

then (4.71) and (4.72) become

$$\dot{y}_1 = [-36 \lambda_2 x_2] y_1 + [-18 x_2^2] y_2 \quad y_1(0.1) = 1.0 \quad (4.73)$$

$$\dot{y}_2 = [-200 + 18 \lambda_2^2] y_1 + [36 \lambda_2 x_2] y_2 \quad y_2(0.1) = 0.0 \quad (4.74)$$

The iterative technique works as follows. A guess is made for  $x_2(0.1)$  in equation (4.65). Equations (4.65), (4.66), (4.73) and (4.74) are integrated backwards in time, from  $t = 0.1$  to  $t = 0$ . A correction is made for the new guess  $x_2(0.1)$ .

$$\Delta x_2(0.1) = \frac{1}{y_1(0)} [1.0 - x_2(0)] \quad (4.75)$$

Convergence using this technique is fast and is obtained in about 5 iterations.

#### Results

For the original problem, the performance index obtained was (see equation (4.35))

$$x_0(0.1) = 10.0 \quad (4.76)$$

The performance index obtained for the relaxed problem was

$$x_0(0.1) = 2.75 \quad (4.77)$$

The last figure shows a significant improvement of the performance of the relaxed problem over the performance of the original problem. For the original problem it is possible to come close to the optimal performance index of the relaxed problem by applying a sub-optimal control to the original problem. When such a control was applied to the original problem using 100 switchings a performance index of

$$x_0(0.1) = 2.82$$

was obtained. The final value of  $x_3$  was

$$x_3(0.1) = 0.6 \cdot 10^{-3}$$

The sub-optimal control shows a great improvement compared to the original performance index for the original problem, and a degradation of only 2.5% compared to the ideal performance index of the relaxed system.

When only 10 switchings were allowed for the sub-optimal control, a performance index of

$$x_0(0.1) = 3.03$$

was obtained. The final value for  $x_3$  was  $x_3(0.1) = 0.032$ .

Figure 4.1 shows the relaxed controls  $u_1^*(t)$  and  $u_2^*(t)$ , and the sub-optimal control for the original problem, consisting of only 10 switchings. Figure 4.2 shows a comparison between  $x_2(t)$  obtained for the relaxed problem and  $x_2(t)$  obtained for the original problem with the sub-optimal control that is shown in Figure 4.1.

#### Game Theory Approach to the Problem

It was assumed before that the error in the initial condition,  $x_2(0)$ , was 1. In general the error is not known and it is desirable to minimize sensitivity with respect to an unknown initial perturbation. The following question arises. Is it possible to find the optimal control for the worst error,  $x_2(0)$ , when  $x_2(0)$  is limited, say between 0 and +1?

$$0 \leq x_2(0) \leq 1$$

For this example three values of possible errors were assumed.

- I.  $x_2(0) = 1.0$
- II.  $x_2(0) = 0.5$
- III.  $x_2(0) = 0.0$

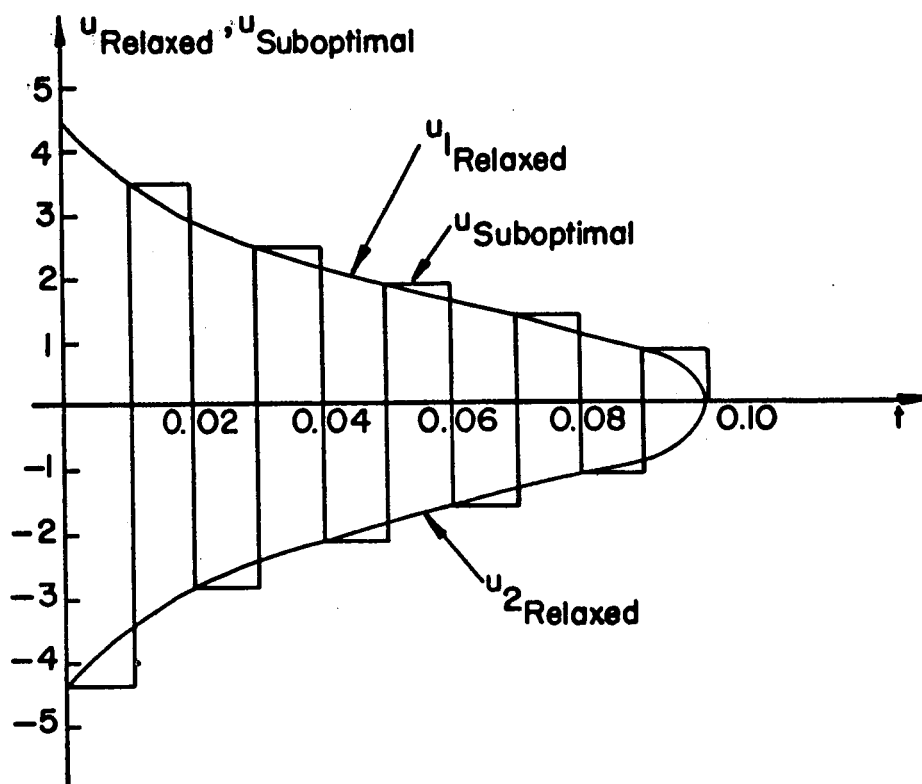


FIGURE 4.1 RELAXED CONTROL AND SUB-OPTIMAL CONTROL FOR EXAMPLE 4.2.

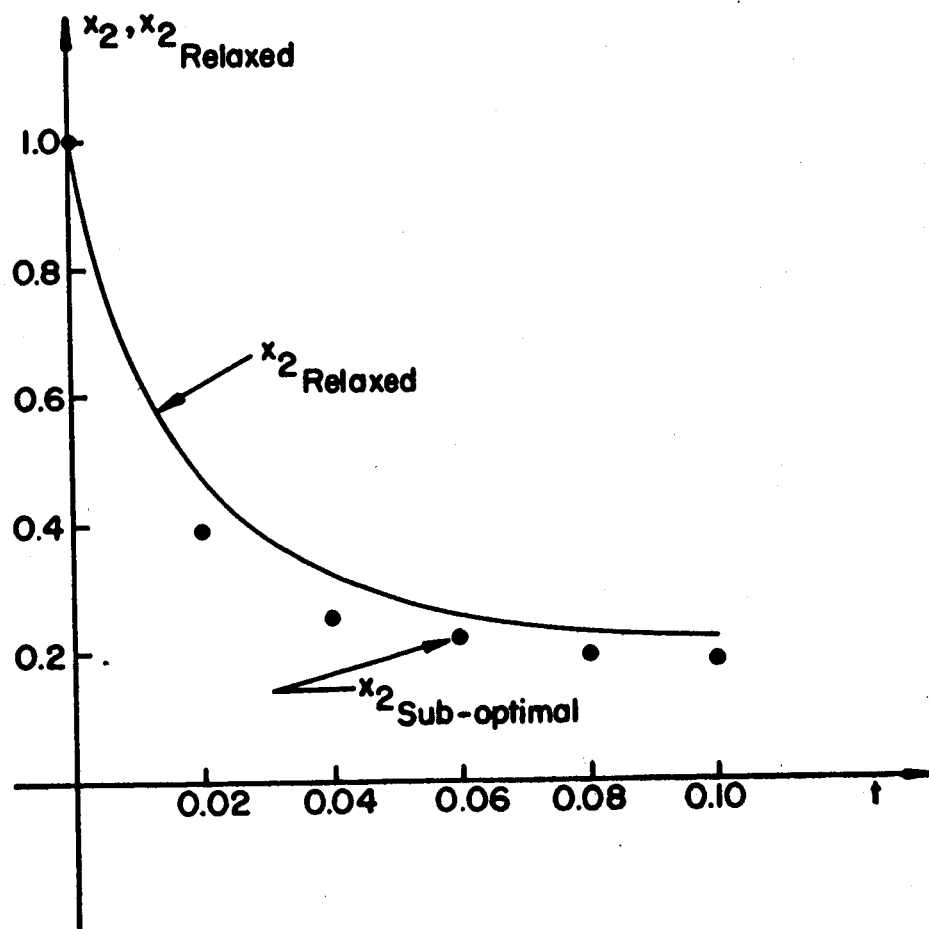


FIGURE 4.2 COMPARISON OF  $x_{2\_Relaxed}$  AND  $x_2$   
FOR 10 SWITCHINGS.

For every one of the three cases, the corresponding optimal control was found. Each one of the three optimal controls was applied three times to the relaxed system for  $x_2(0) = 1.0$ ,  $x_2(0) = 0.5$  and  $x_2(0) = 0.0$ . Thus the experiment resulted in 9 different values of the performance index. These were tabulated in Table 4.3. The first row

Table 4.3 Minimax Solution for Example 4.2

	$x_2(0)=1.0$	$x_2(0)=0.5$	$x_2(0)=0.0$	<div style="text-align: center;">max</div>
u computed for $x_2(0)=1.0$	2.75	1.53	1.12	2.75 min-max
u computed for $x_2(0)=0.5$	3.46	1.15	0.39	3.46
u computed for $x_2(0)=0.0$	10.0	2.5	0.0	10.0
<div style="text-align: center;">min</div>	2.75 max-min	1.15	0.0	

in Table 4.3 consists of the values of the performance index where the optimal control for case I was applied to the relaxed system with  $x_2(0) = 1.0$ ,  $x_2(0) = 0.5$  and  $x_2(0) = 0.0$  respectively. For the second row the control computed for case II was used, and for the third row the control computed for case III was used. When min-max and max-min values are computed for the rows and the columns of Table 4.3 it is clear that

$$\min_u \max_{x_2(0)} \text{PI} = \max_{x_2(0)} \min_u \text{PI} = \text{minimax PI}$$

The corresponding control is the optimal control computed for case I.

Thus in the sense of game theory,  $x_2(0) = 1.0$  is the worst initial perturbation out of the three possibilities  $x_2(0) = 1.0$ ,  $x_2(0) = 0.5$  and  $x_2(0) = 0.0$ , and the corresponding control is the best control.

Thus, if it is known that the possible perturbations for the initial condition,  $x_2(0)$ , can be either  $x_2(0) = 1.0$ ,  $x_2(0) = 0.5$  or  $x_2(0) = 0.0$ , then one should use the control computed for  $x_2(0) = 1.0$  and in the worst case the performance index for the relaxed problem will not be greater than 2.75.

## CHAPTER FIVE

### SUMMARY AND CONCLUSIONS

#### 5.1 Summary

Relaxed Variational Techniques were applied to a minimum sensitivity control problem. The problem does indeed possess an optimal solution but it is possible, at the sacrifice of not satisfying the final conditions exactly, to obtain a considerably better performance using a sub-optimal control. The sub-optimal control is constructed using the optimal controls of the relaxed problem. An example of minimum sensitivity was shown demonstrating the application of the relaxed controls to the construction of the sub-optimal control.

It was demonstrated by an example that a sub-optimal chattering control obtained for a minimum time problem can be made a function of the states of the system and thus minimum sensitivity to perturbation in initial states is achieved as well as minimum time.

A relation was shown to exist between relaxed problems and singular control. It was shown that a problem that does not possess an optimal solution has a singular relaxed solution.

#### 5.2 Suggestions for Further Work

It was shown that a problem that does not possess an optimal

solution has a relaxed singular solution. Thus, in order to find the sub-optimal control it is necessary to solve a singular control problem. It seems therefore important to find efficient numerical techniques suitable for solving optimal control problems that have singular solutions.

In example 2.2 we saw that by a proper choice of the form of the performance index one can get a problem that has an optimal solution (if the performance index is quadratic in  $x$  and  $u$ ), a singular solution (if  $u$  does not appear in the performance index) or no solution at all (as in example 2.2). It seems likely that singular problems are a borderline case between problems that possess optimal solutions and problems that do not have optimal solutions. Further research is needed in this area.

Sensitivity in example 4.2 was defined for a specific case. It would be an important contribution to find a control that would be optimal for arbitrary initial perturbation.

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